9.1. Almost complex structure

An almost complex structure J on a manifold M^m is a $(1,1)$-tensor field with the following property: if for every $p \in M$ we denote by $J_p : T_pM \to T_pM$ the linear map associated with J (recall Theorem T.3), then

$$J_p \circ J_p = -\mathrm{id}_{T_pM}.$$

Prove that every complex manifold admits an almost complex structure.

Hint: Composed with the differential of a complex chart $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$, J_p amounts to the multiplication by i.

Solution. Let $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$ be a chart with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. As suggested by the hint, we define J locally by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad \text{for } i = 1, \ldots, n.$$

It follows that $J_p \circ J_p = -\mathrm{id}_{T_pM}$. It remains to show that J is (globally) well-defined. Let $\psi : V \to \psi(V) \subset \mathbb{C}^n$ be another complex chart on M such that $U \cap V \neq \emptyset$, and denote the coordinates on V by $(u_1, v_1, \ldots, u_n, v_n)$, then

$$\frac{\partial}{\partial x_k} = \sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial v_i}, \quad \frac{\partial}{\partial y_k} = \sum_i \frac{\partial u_i}{\partial y_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial y_k}\frac{\partial}{\partial v_i}.$$

Since $\psi \circ \varphi^{-1}$ is biholomorphic, the Cauchy-Riemann equations imply

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial v_i}{\partial y_k}, \quad \frac{\partial u_i}{\partial y_k} = -\frac{\partial v_i}{\partial x_k}.$$

Denote by J' the corresponding map defined on V with respect to ψ, then

$$\begin{aligned}
J'\left(\frac{\partial}{\partial x_k}\right) &= J'\left(\sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial v_i}\right) \\
&= \sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial v_i} - \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial u_i} \\
&= \sum_i \frac{\partial v_i}{\partial y_k}\frac{\partial}{\partial v_i} + \frac{\partial u_i}{\partial y_k}\frac{\partial}{\partial u_i} \\
&= \frac{\partial}{\partial y_k}
\end{aligned}$$

and similarly $J'\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$. This shows that J and J' coincide on $U \cap V$. $\square$

9.2. Kähler manifolds

Let M be a complex manifold with an almost complex structure $J \in \Gamma(T_{1,1}M)$ (as in Exercise 1). Suppose that M is endowed with an hermitian metric, that is, $g_p(J_p v, J_p w) = g_p(v, w)$ for all $p \in M$ and $v, w \in T_pM$. Show that

$$\omega(X, Y) := g(X, JY)$$

defines a 2-form $\omega \in \Omega^2(M)$, which is closed if and only if J is parallel (i.e. $DJ = D^{1,1}J \equiv 0$).

Solution. By definition ω is a $(0,2)$-tensor field in $\Gamma(T_{0,2}M)$. We still have to show that it is antisymmetric and for that we will use that $J_p^2 = -\mathrm{id}M_p$. For $X, Y \in \Gamma(TM)$ $\omega(X,Y) = g(X, JY) = g\left(JX, J^2Y\right) = -g(JX, Y) = -g(Y, JX) = -\omega(Y, X)$, thus $\omega \in \Omega^2(M)$. In order to prove the second statement, we'll prove the following two identities

$$\begin{aligned} d\omega(X,Y,Z) &= g\left(X, \left(D_Z J\right) Y\right) + g\left(Y, \left(D_X J\right) Z\right) + g\left(Z, \left(D_Y J\right) X\right) \\ 2g\left(D_X(JY), Z\right) &= d\omega(X, JY, JZ) - d\omega(X, Y, Z) \end{aligned}$$

Let X, Y, Z, JX, JY, JZ be coordinate vector fields on a chart of M, in particular they commute. Then (see Theorem 11.3 of Differential Geometry I)

$$\begin{aligned} d\omega(X,Y,Z) &= X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) \\ &= Xg(Y, JZ) - Yg(X, JZ) + Zg(X, JY) \\ &= Xg(Y, JZ) + Yg(Z, JX) + Zg(X, JY) \end{aligned}$$

Thus by the compatibility of the Levi-Civita connection with g and the product rule $D_X(JZ) = \left(D_X J\right) Z + J\left(D_X Z\right)$ for tensor derivations (similarly for the other tuples), we compute

$$\begin{aligned} d\omega(X,Y,Z) =& Xg(Y, JZ) + Yg(Z, JX) + Zg(X, JY) \\ =& g\left(D_X Y, JZ\right) + g\left(Y, D_X(JZ)\right) + g\left(D_Y Z, JX\right) \\ & + g\left(Z, D_Y(JX)\right) + g\left(D_Z X, JY\right) + g\left(X, D_Z(JY)\right) \\ =& g\left(D_X Y, JZ\right) + g\left(D_Y Z, JX\right) + g\left(D_Z X, JY\right) \\ & + g\left(Y, JD_X Z\right) + g\left(Z, JD_Y X\right) + g\left(X, JD_Z Y\right) \\ & + g\left(Y, \left(D_X J\right) Z\right) + g\left(Z, \left(D_Y J\right) X\right) + g\left(X, \left(D_Z J\right) Y\right) \\ =& g\left(Y, \left(D_X J\right) Z\right) + g\left(Z, \left(D_Y J\right) X\right) + g\left(X, \left(D_Z J\right) Y\right), \end{aligned}$$

where in the last step we have used that g is hermitian and the vector fields commute. This proves the first identity.

For the second identity first note that $g\left(\left(D_X J\right) Y, Z\right)=g\left(D_X(J Y), Z\right)-g\left(J D_X Y, Z\right)=g\left(D_X(J Y), Z\right)+g\left(D_X Y, J Z\right)$. Now, by the Koszul formula we have

$$\begin{aligned}
2 g\left(D_X(J Y), Z\right) & =X g(J Y, Z)+J Y g(X, Z)-Z g(X, J Y) \\
& =X \omega(Z, Y)-J Y g(X, J J Z)-Z \omega(X, Y) \\
& =-X \omega(Y, Z)-J Y \omega(X, J Z)-Z \omega(X, Y)
\end{aligned}$$

and

$$\begin{aligned}
2 g\left(D_X Y, J Z\right) & =X g(Y, J Z)+Y g(X, J Z)-J Z g(X, Y) \\
& =-X g(J Z, J J Y)+Y \omega(X, Z)+J Z g(X, J J Y) \\
& =-X \omega(J Z, J Y)+Y \omega(X, Z)+J Z \omega(X, J Y)
\end{aligned}$$

By summing the two expression we obtain the second identity. □